A CURIOUS EQUATION INVOLVING THE ∞-LAPLACIAN

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Abstract. We prove the uniqueness for viscosity solutions of the differential equation

$$\sum u_{x_i}u_{x_j}u_{x_ix_j} + |\nabla u|^2 \ln|\nabla u| \langle \nabla u, \nabla \ln p\rangle = 0.$$

A variant of the Harnack inequality is derived. The equation comes from the problem of finding

$$\min_u \max_x (|\nabla u(x)|^{p(x)}).$$

The positive exponent $p(x)$ is a continuously differentiable function.

1. Introduction

The object of our study is the differential equation

$$\sum_{i,j=1}^{n} u_{x_i}u_{x_j}u_{x_ix_j} + |\nabla u|^2 \ln|\nabla u| \langle \nabla u, \nabla \ln p\rangle = 0. \tag{1.1}$$

in a bounded domain Ω in $\mathbb{R}^n$. The main result is the uniqueness for viscosity solutions, Theorem 1.2. We also have a Harnack inequality in Section 4.

In the case of the constant function $p(x) = p$, the last term is not present, and the equation is well-known. During the last fifteen years there has been a lot of research done for the ∞-Laplace equation

$$\Delta_\infty u \equiv \sum_{i,j=1}^{n} u_{x_i}u_{x_j}u_{x_ix_j} = 0. \tag{1.2}$$

This is the limit, as $p \to \infty$, of the Euler–Lagrange equations

$$\Delta_p u \equiv |\nabla u|^{p-2} \Delta u + (p-2)|\nabla u|^{p-4} \Delta_\infty u = 0 \tag{1.3}$$

for the variational integrals

$$\int_\Omega |\nabla u(x)|^p \, dx.$$

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Thus it appears that the equation $\Delta_\infty u = 0$ is formally the Euler–Lagrange equation of the "variational integral"

$$I(u) = \|\nabla u\|_{L^\infty(\Omega)} = \operatorname*{ess\,sup}_{\Omega} |\nabla u| .$$

It is sometimes called Aronsson's Euler equation, after its discoverer, who derived the equation in order to find the best Lipschitz extension of given boundary values, cf. [Aro67]. The equation must be interpreted in the sense of viscosity solutions. We assume that the reader is familiar with the basic facts of this fascinating theory, cf. [CIL92, Cra97, Koi04].

In his remarkable work [Jen93] R. Jensen succeeded in proving the uniqueness of the viscosity solutions to (1.2). We will closely follow Jensen's construction of auxiliary equations, when we come to our uniqueness proof below.

Let us return to the equation (1.1). The problem about

$$\min_u \max_x (|\nabla u(x)|^{p(x)})$$

with a variable exponent can be reached via the variational integrals

$$\left\{ \int_\Omega |\nabla u(x)|^{kp(x)} \frac{\mathrm{d}x}{kp(x)} \right\}^{\frac{1}{k}} \tag{1.4}$$

as $k \to \infty$. Such integrals were first considered by Zhikov, cf. [Zhi86]. The Euler–Lagrange equation is

$$\begin{aligned} \Delta_{kp(x)} u \equiv |\nabla u|^{kp(x)-2} \Delta u + (kp(x) - 2) |\nabla u|^{kp(x)-4} \Delta_\infty u \\ + |\nabla u|^{kp(x)-2} \ln |\nabla u| \langle \nabla u, \nabla kp(x) \rangle = 0. \end{aligned}$$

Notice the extra term with ∇p. Its formal limit as $k \to \infty$ is the equation

$$\Delta_{\infty(x)} u \equiv \Delta_\infty u + |\nabla u|^2 \ln |\nabla u| \langle \nabla u, \nabla \ln p \rangle = 0, \tag{1.5}$$

and the reader can recognize (1.1).

Thus the operator Δ_∞ is replaced by the new operator $\Delta_{\infty(x)}$, the ∞-Laplacian with variable exponent. Again, the interpretation is delicate, since now ∇p is needed pointwise. To be on the safe side, we therefore assume that $p \in C^1(\Omega) \cap W^{1,\infty}(\Omega)$, $p(x) > 1$, and that Ω is a bounded domain in $\mathbb{R}^n$. Then viscosity solutions to (1.1) can be defined in the standard way.

Definition 1.1. We say that a lower semicontinuous function $v : \Omega \to (-\infty, \infty]$ is a *viscosity supersolution* if, whenever $x_0 \in \Omega$ and $\varphi \in C^2(\Omega)$ are such that

(1) $\varphi(x_0) = u(x_0)$, and
(2) $\varphi(x) < v(x)$, when $x \neq x_0$,

we have

$$\Delta_{\infty(x_0)} \varphi(x_0) \leq 0.$$

The *viscosity subsolutions* have a similar definition; they are upper semicontinuous, the test functions touch from above and the differential inequality is reversed. Finally, a *viscosity solution* is a function that is both a viscosity supersolution and viscosity subsolution. A very simple example is

$$u(x) = |x|\,;$$

it is a viscosity subsolution for all exponents $p(x)$.

The boundary values are prescribed by a Lipschitz continuous function $f : \partial\Omega \to \mathbb{R}$. It can be extended to the whole space with the same constant, say

$$|f(x) - f(y)| \leq L\,|x - y|\,, \quad x, y \in \mathbb{R}^n.$$

According to Rademacher's theorem f is differentiable a.e. and $\|\nabla f\|_\infty \leq L$. After the extension, one has $f \in W^{1,\infty}(\mathbb{R}^n)$.

The following existence and uniqueness result holds for the Dirichlet boundary value problem.

Theorem 1.2. *Given a Lipschitz continuous function $f : \partial\Omega \to \mathbb{R}$, there exists a unique viscosity solution $u \in C(\overline{\Omega})$ with boundary values f. Moreover, $u \in W^{1,\infty}(\Omega)$, and $\|\nabla u\|_{L^\infty(\Omega)}$ has a bound depending only on the Lipschitz constant of f.*

The main part of the proof is in Lemma 3.1 and Lemma 2.2. The result below can be extracted from our constructions.

Theorem 1.3 (Comparison principle)**.** *If u is a viscosity subsolution and v a viscosity supersolution to* (1.1) *in Ω, then $v \geq u$ on $\partial\Omega$ implies that $v \geq u$ in Ω.*

Passing a Caccioppoli estimate for the minimizers of the integrals (1.4) to the limit, we obtain the following form of Harnack's inequality. This is similar to the one proved by Alkhutov [Alk97].

Theorem 1.4 (Harnack's inequality)**.** *Let u be a nonnegative viscosity solution of* (1.1)*. Then the inequality*

$$\sup_{x \in B_R} u(x) \leq C \left(\inf_{x \in B_R} u(x) + R \right)$$

holds, with the constant depending on the supremum of u taken over $B_{2R} \subset \Omega$.

2. The auxiliary equations

Following a device of Jensen in [Jen93], we introduce two auxiliary equations with a positive parameter ε. The situation will be

$$\max\{\varepsilon - |\nabla v|\,, \Delta_{\infty(x)} v\} = 0 \qquad \textit{Upper equation} \tag{2.1}$$

$$\Delta_{\infty(x)} h = 0 \qquad \textit{Equation} \tag{2.2}$$

$$\min\{|\nabla u| - \varepsilon, \Delta_{\infty(x)} u\} = 0 \qquad \textit{Lower equation} \tag{2.3}$$

If the solutions have the same boundary values, it turns out that $u \leq h \leq v$. We say that a continuous function $v : \Omega \to \mathbb{R}$ is a viscosity supersolution of the upper equation, if whenever $x_0 \in \Omega$ and $\varphi \in C^2(\Omega)$ are such that

(1) $\varphi(x_0) = v(x_0)$,
(2) $\varphi(x) < v(x)$, when $x \neq x_0$

then we have

$$\varepsilon - |\nabla\varphi(x_0)| \leq 0 \text{ and } \Delta_{\infty(x_0)}\varphi(x_0) \leq 0.$$

Notice that the differential operator is evaluated only at the touching point x_0. The other definitions are analogous; the viscosity subsolutions of the lower equation are also used later.

The existence of solutions is proved through the following variational procedure. The equation

$$\Delta_{kp(x)}u = -\varepsilon^{kp(x)-1},$$

or in its weak form

$$\int_\Omega \langle |\nabla u|^{kp(x)-2}\,\nabla u, \nabla\eta\rangle \,\mathrm{d}x = \int_\Omega \varepsilon^{kp(x)-1}\eta\,\mathrm{d}x, \tag{2.4}$$

when $\eta \in C_0^\infty(\Omega)$, is the Euler–Lagrange equation of the variational integral

$$\int_\Omega |\nabla u|^{kp(x)}\,\frac{\mathrm{d}x}{kp(x)} - \int_\Omega \varepsilon^{kp(x)-1}\,\mathrm{d}x. \tag{2.5}$$

As $k \to \infty$, we get the *upper equation*

$$\max\{\varepsilon - |\nabla v|\,, \Delta_{\infty(x)}v\} = 0. \tag{2.6}$$

Suppose now that $u_k \in C(\overline{\Omega}) \cap W^{1,kp(x)}(\Omega)$, $u_k = f$ on $\partial\Omega$, is the minimizer of the above variational integral. Then it satisfies the weak equation and a standard procedure shows that it is a viscosity solution, see for example [Jen93] about the method. It follows from the minimizing property that

$$\int_\Omega |\nabla v_k|^{kp(x)}\,\frac{\mathrm{d}x}{kp(x)} \leq \int_\Omega |\nabla f|^{kp(x)}\,\frac{\mathrm{d}x}{kp(x)} + \int_\Omega \varepsilon^{kp(x)-1}(v_k - f)\,\mathrm{d}x,$$

since f is admissible.We can see that $v_k \to v \in C(\overline{\Omega}) \cap W^{1,\infty}(\Omega)$ uniformly in Ω, at least for a subsequence. We have

$$\left\| |\nabla v|^{p(x)} \right\|_\infty \leq \left\| |\nabla f|^{p(x)} \right\|_\infty + \left\| \varepsilon^{p(x)} \right\|_\infty.$$

It is again a standard procedure to verify that this v is a viscosity supersolution of the upper equation, cf. [Jen93]. (It is also a viscosity subsolution.) We call v a *variational solution*, because it is a limit of minimizers of variational integrals. We record an immediate estimate.

Lemma 2.1. *A variational solution of the upper equation satisfies*

$$\|\nabla v\|_\infty \leq K,$$

where K depends only on the Lipschitz constant of the boundary values.

For the lower auxiliary equation

$$\min\{|\nabla u| - \varepsilon, \Delta_{\infty(x)} u\} = 0 \tag{2.7}$$

the various stages are

$$\Delta_{kp(x)} u = \varepsilon^{kp(x)-1},$$

$$\int_\Omega \langle |\nabla u|^{kp(x)-2} \nabla u, \nabla \eta \rangle \,\mathrm{d}x = -\int_\Omega \varepsilon^{kp(x)-1} \eta \,\mathrm{d}x,$$

$$\int_\Omega |\nabla u|^{kp(x)} \frac{\mathrm{d}x}{kp(x)} + \int_\Omega \varepsilon^{kp(x)-1} \,\mathrm{d}x.$$

The situation is analogous to the previous case, but now the subsolutions count.

We have to construct solutions of the auxiliary equations that are close for small values of ε. Let u_k^-, u_k, and u_k^+ be the weak solutions of the lower equation (2.3), the equation (2.2) and the upper equation (2.1), all with the same boundary values f. Then

$$u_k^- \leq u_k \leq u_k^+$$

by comparison. The weak solutions are viscosity solutions of their respective equations. Select a subsequence of indices so that all three converge, say $u_k^- \to u^-$, $u_k \to h$, and $u_k^+ \to u^+$. Thus

$$\operatorname{div}(\left|\nabla u_k^+\right|^{kp(x)-2} \nabla u_k^+) = -\varepsilon^{kp(x)-1}$$

$$\operatorname{div}(\left|\nabla u_k^-\right|^{kp(x)-2} \nabla u_k^-) = +\varepsilon^{kp(x)-1},$$

and, using $u_k^+ - u_k^-$ as a test function in the weak formulation of the equations and subtracting these, we obtain

$$\int_\Omega \langle \left|\nabla u_k^+\right|^{kp(x)-2} \nabla u_k^+ - \left|\nabla u_k^-\right|^{kp(x)-2} \nabla u_k^-, \nabla u_k^+ - \nabla u_k^- \rangle \,\mathrm{d}x$$

$$= \int_\Omega \varepsilon^{kp(x)-2} (u_k^+ - u_k^-) \,\mathrm{d}x.$$

With the aid of the elementary inequality

$$\langle |b|^{q-2} b - |a|^{q-2} a, b-a \rangle \geq 2^{2-q} |b-a|^q$$

for vectors b, a and $q \geq 2$, we obtain

$$4 \int_\Omega \left| \frac{\nabla u_k^+ - \nabla u_k^-}{2} \right|^{kp(x)} \mathrm{d}x \leq \frac{1}{\varepsilon} \int_\Omega \varepsilon^{kp(x)} (u_k^+ - u_k^-) \,\mathrm{d}x.$$

Extracting the kth roots, we conclude that

$$\operatorname{ess\,sup} \left| \frac{\nabla u^+ - \nabla u^-}{2} \right|^{p(x)} \leq \sup(\varepsilon^{p(x)}).$$

Keeping $\varepsilon < 1$, we arrive at the estimates

$$\begin{aligned} \left\|\nabla u^+ - \nabla u^-\right\|_\infty &\leq C'\varepsilon^\kappa \\ \left\|u^+ - u^-\right\|_\infty &\leq C\varepsilon^\kappa \end{aligned}$$

where κ depends only on the bounds on $p(x)$. The obtained functions u^+, u^- and h are viscosity solutions of their equations.

We have the result

$$u^- \leq h \leq u^+ \leq u^- + \mathcal{O}(\varepsilon^\kappa)$$

for solutions u^- (lower equation), h (the equation), and u^+ (upper equation) coming from the variational procedure. This does not yet prove that variational solutions are unique. The possibility that another subsequence yields three new ordered solutions is difficult to exclude. (At least it can be arranged so that the same h will do for all ε, though the constructed u^-, u^+ depend on ε.)

Lemma 2.2. *If $u \in C(\overline{\Omega})$ is an arbitrary viscosity solution of the equation, $u = f$ on $\partial\Omega$, then*

$$u^- \leq u \leq u^+,$$

where u^-, u^+ are the constructed variational solutions of the auxiliary equations.

This lemma, which will be proved in Section 3, implies that

$$|h - u| \leq \mathcal{O}(\varepsilon^\kappa).$$

Since u is independent of ε, it is unique; use

$$|u_1 - u_2| \leq |h - u_1| + |h - u_2| \leq \mathcal{O}(\varepsilon^\kappa)$$

and let $\varepsilon \to 0$ to see that two viscosity solutions u_1, u_2 coincide. Thus Theorem 1.2 follows. This also shows that u is a variational solution, since $u = h$.

3. Proof of the comparison principle

Recall the variational solutions u^+ and u^- of the auxiliary equations in Section 2. They satisfy the inequalities

$$\varepsilon \leq \left|\nabla u^\pm\right| \leq K.$$

The crucial part of the proof is in the following lemma.

Lemma 3.1. *If $u \in C(\overline{\Omega})$ is a viscosity subsolution of the equation* (1.1)*, and if $u \leq f = u^+$ on $\partial\Omega$, then $u \leq u^+$ in Ω.*

The analogous comparison holds for viscosity supersolutions lying above u^-.

Proof. By adding a constant we may assume that $u^+ > 0$. Write $v = u^+$ for simplicity. We claim that $v \geq u$. We use the *antithesis*

$$\max_{\Omega}(u - v) > \max_{\partial\Omega}(u - v).$$

We will construct a strict supersolution $w = g(v)$ of the upper equation such that also

$$\max_{\Omega}(u - w) > \max_{\partial\Omega}(u - w), \tag{3.1}$$

and

$$\Delta_{\infty(x)} w \leq -\mu < 0$$

in Ω. This will lead to a contradiction.

In fact, we will use the expedient approximation

$$g(t) = \frac{1}{\alpha} \log(1 + A(e^{\alpha t} - 1))$$

of the identity, which was studied in [JLM99]. Here $A > 1$ and $\alpha > 0$. Now

$$\begin{aligned} 0 < g(t) - t < \frac{A-1}{\alpha}, \\ 0 < g'(t) - 1 < A - 1, \end{aligned}$$

assuming that $t \geq 0$. Further

$$\frac{g''(t)}{g'(t)} = -\alpha[g'(t) - 1] = -\frac{\alpha(A-1)}{1 + A(e^{\alpha t} - 1)}, \tag{3.2}$$

and

$$0 \leq \log(g'(t)) = \log[1 + (g'(t) - 1)] \leq g'(t) - 1, \tag{3.3}$$

provided that $A < 2$.

To prove the comparison, we need the equation for $w = f(v)$. We have

$$\begin{aligned} w = g(v), \quad w_{x_i} &= g'(v) v_{x_i}, \\ w_{x_i x_j} &= g''(v) v_{x_i} v_{x_j} + g'(v) v_{x_i x_j} \\ \Delta_\infty w &= g'(v)^3 \Delta_\infty v + g'(v)^2 g''(v) \left|\nabla v\right|^4. \end{aligned}$$

Multiplying the upper equation (for supersolutions)

$$\max\{\varepsilon - |\nabla v|, \Delta_{\infty(x)} v\} \leq 0$$

by $g'(v)^3$, we formally obtain that

$$\Delta_\infty w - g'(v)^2 g''(v) \left|\nabla v\right|^4 + \left|\nabla w\right|^2 \ln(|\nabla v|)\langle \nabla w, \nabla \ln p\rangle \leq 0.$$

Writing $\ln|\nabla v| = \ln(|\nabla w|) - \ln(g'(v))$ we obtain the equation

$$\begin{aligned}\Delta_\infty w + |\nabla w|^2 \ln(|\nabla w|)\langle \nabla w, \nabla \ln p\rangle \\ \leq |\nabla w|^3 \left[\frac{g''(v)}{g'(v)}\,|\nabla v| + \ln(g'(v))\,|\nabla \ln p|\right].\end{aligned}$$

We also have $\varepsilon \leq |\nabla v|$. Using (3.2) and (3.3), we can write

$$\begin{aligned}\Delta_\infty w &\leq |\nabla w|^3\,[-\alpha\varepsilon + \|\nabla \ln p\|_\infty]\,\frac{A-1}{1+A(e^{\alpha v}-1)} \\ &\leq \varepsilon^3 g'(v)^4[-\alpha\varepsilon + \|\nabla \ln p\|_\infty](A-1)A^{-1}e^{-\alpha v}.\end{aligned}$$

Given $\varepsilon > 0$, fix $\alpha = \alpha(\varepsilon)$ so large that

$$-\alpha\varepsilon + \|\nabla \ln p\|_\infty \leq -2.$$

Then fix A so close to 1 that

$$0 < w - v = g(v) - v < \frac{A-1}{\alpha} < \delta,$$

where δ is small enough to guarantee (3.1). With these adjustments, the right-hand member is less than the *negative* quantity

$$-\mu = -\varepsilon^3 1^4 (A-1)A^{-1}e^{-\alpha\|v\|_\infty}.$$

The resulting equation is

$$\Delta_{\infty(x)} w \leq -\mu.$$

The described procedure was formal. The reader should replace v by a test function φ touching v from below at a point x_0 and w should be replaced by the test function $\psi = g(\varphi)$, which touches w from below. The inversion $\varphi = g^{-1}(\psi)$ is evident. We have proved that

$$\Delta_{\infty(x_0)}\psi(x_0) \leq -\mu$$

whenever ψ touches w from below at $x_0 \in \Omega$.

We aim at using "the theorem of sums", formulated in terms of the so-called superjets and subjets. For the jets and their closures, we refer to [CIL92, Cra97, Koi04]. Start with doubling the variables:

$$M = \sup_{\substack{x\in\Omega \\ y\in\Omega}} \left(u(x) - w(y) - \frac{j}{2}\,|x-y|^2\right).$$

The maximum is attained at the interior points x_j, y_j (for large indices) and

$$x_j \to \hat{x}, \qquad y_j \to \hat{x},$$

where $\hat{x}$ is an interior point, the same for both sequences. It cannot be on the boundary due to (3.1). (It is known that $j\,|x_j - y_j|^2 \to 0$.) We need the bound

$$j\,|x_j - y_j| \leq C.$$

To obtain it, we reason as follows:

$$u(x_j) - w(y_j) - \frac{j}{2}|x_j - y_j|^2$$
$$\geq u(x_j) - w(x_j) - \frac{j}{2}|x_j - x_j|^2 = u(x_j) - w(x_j),$$

so that

$$\frac{j}{2}|x_j - y_j|^2 \leq w(x_j) - w(y_j) \leq \|\nabla w\|_\infty |x_j - y_j|.$$

Now $\|\nabla w\|_\infty = \|g'(v)\nabla v\|_\infty \leq KA$ according to Lemma 2.1. The bound follows with $C = 2KA$. (This is why v has to be a variational solution!) Further, we need the bound

$$j|x_j - y_j| \geq \varepsilon,$$

which follows from $\nabla w = g'(v)\nabla v$, since $g'(v) \geq 1$ and $|\nabla v| \geq \varepsilon$ in the viscosity sense.

The theorem of sums assures that there exist symmetric $n \times n$-matrices X_j and Y_j such that $X_j \leq Y_j$ and

$$(j(x_j - y_j), X_j) \in \overline{J^{2,+}}u(x_j),$$
$$(j(x_j - y_j), Y_j) \in \overline{J^{2,-}}w(y_j)$$

where $\overline{J^{2,+}}u(x_j)$ and $\overline{J^{2,-}}w(y_j)$ are the closures of the super- and subjets, respectively. We can rewrite the equations as

$$\begin{aligned}
&j^2\langle Y_j(x_j - y_j), x_j - y_j\rangle \\
&\quad + j^3|x_j - y_j|^2 \ln(j|x_j - y_j|)\langle x_j - y_j, \nabla \ln p(y_j)\rangle \leq -\mu, \\
&j^2\langle X_j(x_j - y_j), x_j - y_j\rangle \\
&\quad + j^3|x_j - y_j|^2 \ln(j|x_j - y_j|)\langle x_j - y_j, \nabla \ln p(x_j)\rangle \geq 0, \\
&j|x_j - y_j| \geq \varepsilon, \\
&j|x_j - y_j| \leq C.
\end{aligned}$$

Subtract the equations and move the terms with $\ln p$. Then

$$\begin{aligned}
&j^2\langle (Y_j - X_j)(x_j - y_j), (x_j - y_j)\rangle \\
&\quad \leq -\mu \\
&\qquad + j^3|x_j - y_j|^2 \ln(j|x_j - y_j|)\langle x_j - y_j, \nabla \ln p(x_j) - \nabla \ln p(y_j)\rangle \\
&\quad \leq -\mu + C^3 \ln\left(\frac{C}{\varepsilon}\right)|\nabla \ln p(x_j) - \nabla \ln p(y_j)|.
\end{aligned}$$

The last term approaches zero as $j \to \infty$, because of the continuity. The very first term is non-negative, since $Y_j \geq X_j$. The contradiction

$$0 \leq -\mu + 0$$

arises. Therefore, the antithesis is false, and consequently $u \leq v$. This concludes the proof. $\square$

4. Estimates for solutions

In this section, we prove some simple estimates for the positive viscosity solutions of (1.1). In particular, we show that they satisfy a version of Harnack's inequality, similar to the one for solutions of the $p(x)$-Laplacian, cf. [Alk97].

We start by deriving a Caccioppoli estimate for finite exponents. Let u be a nonnegative minimizer, and set

$$v(x) = u(x) + \varepsilon \zeta(x)^{p(x)} u(x)^{1-p(x)},$$

where $\zeta \in C_0^\infty(\Omega)$ is nonnegative. Then

$$\begin{aligned}\nabla v = &\left(1 - (p-1)\varepsilon \left(\frac{\zeta}{u}\right)^p\right) \nabla u \\ &+ \varepsilon(p-1)\left(\frac{\zeta}{u}\right)^p \left[\frac{p}{p-1}\left(\frac{u}{\zeta}\right)\nabla\zeta + \frac{1}{p-1} u \ln\left(\frac{\zeta}{u}\right)\nabla p\right].\end{aligned}$$

Observe that ∇v is a convex combination of ∇u and the expression in the brackets, provided that $\lambda = (p-1)\varepsilon\left(\frac{\zeta}{u}\right)^p \le 1$. To see that we may choose ε sufficiently small to accomplish this, first consider $u + \delta > 0$ instead of u. Since ε will disappear from the estimate, we may safely let $\delta \to 0$ in the end.

We use the minimizing property of u and the convexity to get

$$\begin{aligned}&\int_\Omega |\nabla u|^p \frac{\mathrm{d}x}{p} \le \int_\Omega |\nabla v|^p \frac{\mathrm{d}x}{p} \\ &\le \int_\Omega \left(1 - \varepsilon(p-1)\left(\frac{\zeta}{u}\right)^p\right)|\nabla u|^p \frac{\mathrm{d}x}{p} \\ &\quad + \int_\Omega \varepsilon(p-1)\left(\frac{\zeta}{u}\right)^p \left|\frac{p}{p-1}\left(\frac{u}{\zeta}\right)\nabla\zeta + \frac{1}{p-1}u\ln\left(\frac{\zeta}{u}\right)\nabla p\right|^p \frac{\mathrm{d}x}{p}.\end{aligned}$$

This simplifies to

$$\int_\Omega (p-1)\zeta^p |\nabla \ln u|^p \frac{\mathrm{d}x}{p} \le \int_\Omega \frac{p^p}{(p-1)^{p-1}} \left|\nabla\zeta + \zeta\ln\left(\frac{\zeta}{u}\right)\frac{\nabla p}{p}\right|^p \frac{\mathrm{d}x}{p}$$

where ε has now disappeared.

Next, we replace $p(x)$ by $kp(x)$; the above inequality then holds for the corresponding minimizers u_k, and we can assume that they converge uniformly to the solution u of (1.1). Then we can take kth roots and let $k \to \infty$. This gives the following lemma.

Lemma 4.1. *Let u be a positive viscosity solution to* (1.1)*, and ζ a positive, compactly supported smooth function. Then*

$$\sup_{x\in\Omega} |\zeta(x)\nabla \ln u(x)|^{p(x)} \le \sup_{x\in\Omega}\left|\nabla\zeta(x) + \zeta(x)\ln\left(\frac{\zeta(x)}{u(x)}\right)\nabla \ln p(x)\right|^{p(x)}.$$

Observe that Lemma 4.1 reduces to a well-known estimate for solutions to the ∞- Laplace equation (1.2) when $p(x)$ is constant.

Harnack's inequality is now a rather simple consequence of Lemma 4.1. Indeed, take a cutoff function ζ compactly supported in $B(x_0, 2R)$ such that $\zeta = 1$ in $B(x_0, R)$, $0 \leq \zeta \leq 1$ and $|\nabla \zeta| \leq 2/R$. Then, for $v = u + R$, we get by the fundamental theorem of calculus that

$$|\ln v(x) - \ln v(y)| \leq \left(1 + \left\| |\zeta \nabla \ln v|^{p(x)} \right\|_{\infty, B(x_0, 2R)}\right) |x - y|$$

for $x, y \in B(x_0, R)$. Observing that $|\ln v| \leq \max\{R^{-1}, \|v\|_{\infty, B(x_0, 2R)}\}$, Lemma 4.1 implies that the right hand side is estimated by

$$C_1 \|v\|_\infty |x - y| + C_2 \frac{|x - y|}{R}.$$

Taking exponents of both sides and replacing v by $u + R$ we get

$$u(x) + R \leq \exp(C_2 |x - y| / R) \exp(C_1 \|u + R\|_\infty |x - y|)(u(y) + R).$$

The Harnack inequality of Theorem 1.4 follows from this, although the estimate is more powerful.

A couple of variants are also obtained by similar reasoning. For instance, one can replace R by R^α for any $\alpha > 0$, the price to pay being that $C_2 = \mathcal{O}(\alpha)$. Similarly, one can have any positive power ε on the supremum of u in the constant, with $C_1 = \mathcal{O}(1/\varepsilon)$.

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